Problem Set 4: due Tuesday, February 27, 2018

Useful references:
Frank Shu; "Gas Dynamics"
P. Drazin and W. Reid; "Hydrodynamic Stability"
And, of course, Landau and Lifshitz

1) Tapas

Each of these questions does not require more than a few lines of calculation. *Don't* make them longer than they need be. Do state your reasoning clearly. Try to do these closed book.

a) What is the width of the laminar boundary layer at the bottom of a viscous fluid rotating at Ω ?

b) What is the width of a laminar boundary layer of a stagnation flow?

c) What shaped eddy is most effective at Rayleigh–Bernard convection when $\Omega^2 >> g\alpha\beta$?

d) How will the strength of a vortex tube evolve when the fluid density within rises linearly in time?

e) A sphere of radius R moves thru an inviscid fluid line (i.e. obeys Euler, not Navïer–Stokes) with a free surface. The sphere moves at V, at various depths d. What happens? Estimate the drag on the sphere.

2) Calculate the flow in a boundary layer in a converging channel between two non-parallel plates. See Landau sections 23, 39. Explain your similarity solution carefully.

3) Now consider a rotating fluid which is also compressible and *self-gravitating*. For the latter, include a body force $\underline{f} = -\nabla\phi$ where:

$$\nabla^2\phi = 4\pi G\rho$$

Take $\underline{\Omega} = \Omega\hat{z}$, as usual.

a) For $\underline{k} = k\hat{z}$, show

$$\omega^2 = k^2 c_s^2 - 4\pi G\rho_0 \ .$$

Welcome to the Jeans instability! What might be the significance of the marginally stable length scale?

b) For $\underline{k} = k\hat{x}$, show:

$$\omega^2 = k^2 c_s^2 - 4\pi G \rho_0 + 4\Omega^2 .$$

When are all modes stable?

c) Why might this result be of interest in the context of galaxy structure?

d) How might Jeans instabilities evolve nonlinearly? Why might cooling effects be important here?

4) Consider a sheared flow $\underline{v}(z)\hat{x}$ in a stably stratified fluid with $\underline{g} = -g\hat{z}$.

a) Derive the 2D wave eigenmode equation, called the Taylor–Goldstein equation. This extends the Rayleigh equation from the last problem of Set 3.

b) As before, take $\omega = \omega_r + i\gamma$, and substitute $H = \phi / (V - c)^{1/2}$ where c is the along-stream phase velocity. Multiply the re-scaled Taylor–Goldstein equation by H^* and derive a quadratic form.

c) Show for $\gamma > 0$, the *Richardson number* must satisfy:

$JN^2/V'^2 < 1/4$, for shear-driven instability.

Here $\frac{JN^2}{V'^2} = \frac{-g}{\rho}\frac{d\rho}{dz}/V'^2$, is the Richardson number.

d) What is the physics of the Richardson number criterion? What does it mean, in simple terms?

e) Extra Credit: Can you give a grunt-and-hand gesture derivation of the Richardson number criterion, based on simple physical ideas of competition of energetics?

5) <u>Calculate</u> the boundary layer thickness and flow field near a semi-infinite plate, assuming laminar flow. This is the Blasius problem. Follow the self-similarity approach discussed in Landau and Lifshitz. Get as far as you can with the numerical factors.

6a) Using Oseen's approximation, calculate the shape, flow field and pressure field for the laminar wake, assuming conical symmetry, as discussed in class.

b) Now, do the same for a turbulent wake, exploiting the idea of a turbulent viscosity, as introduced in class.

c) How far behind the object does the wake transition from turbulent to laminar?